Received: *February 24, 1999*, Accepted: *March 29, 1999*

A new approach to axial vector model calculations II

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Abstract: We further develop the new approach, proposed in part I, to computing the heat kernel associated with a Fermion coupled to vector and axial vector fields. We first use the path integral representation obtained for the heat kernel trace in a vector-axialvector background to derive a Bern-Kosower type master formula for the one-loop amplitude with M vectors and N axialvectors, valid in any even space-time dimension. For the massless case we then generalize this approach to the full off-diagonal heat kernel. In the $D = 4$ case the SO(4) structure of the theory can be broken down to $\mathrm{SU}(2)\times\mathrm{SU}(2)$ by use of the 't Hooft symbols. Various techniques for explicitly evaluating the spin part of the path integral are developed and compared. We also extend the method to external fermions, and to the inclusion of isospin. On the field theory side, we obtain an extension of the second order formalism for fermion QED to an abelian vector-axialvector theory.

Keywords: Field Theories in Lower Dimensions, Anomalies in Field and String Theories.

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1. Introduction

It has been known for a long time that first-quantized particle path integrals are sometimes a useful alternative to standard second-quantized methods in quantum field theory. Introduced by Feynman himself at the dawn of relativistic quantum field theory [1], particle path integral techniques have found applications to a variety of problems in quantum field theory, ranging from QED [2, 3] to anomalies [4, 5, 6], meson-nucleon theory [7], and non-perturbative QCD [8, 9]. Only in recent years, however, was this approach also seriously considered as a competitor to standard Feynman diagram methods for the calculation of the basic quantities in perturbation theory such as one-loop scattering amplitudes. The renewed interest in this approach is largely due to the work of Bern and Kosower, who derived new rules for the construction of one-loop QED/QCD and supergravity amplitudes by analyzing the infinite string tension limits of the corresponding amplitudes in appropriate string models [10]. Those "Bern-Kosower Rules" were applied successfully to the calculation of five-gluon amplitudes [11] as well as four-graviton amplitudes [12]. Strassler [13] succeeded in rederiving those same "Bern-Kosower Rules", without the use of string theory, by representing the QED/QCD one-loop effective action in terms of suitable first-quantized path integrals, and evaluating them in a way analogous to string theory [14].[1] This approach to the Bern-Kosower formalism has been further developed by various authors [15, 16, 17, 18, 19, 20, 21]. It turned out to be very effective for the calculation of photon amplitudes in QED, in particular for problems involving constant external fields [22, 23, 24, 25, 26].

A similar formalism was developed in an independent line of work starting from the heat kernel representation of quantum field theory propagators [27, 28, 29]. For a discussion of the differences between both approaches see [30].

Until very recently particle path integral representations for one-loop effective actions were known essentially only for scalar and for (vector-)gauge theories. Generalizations to Yukawa and axial couplings were constructed heuristically in [31], and more rigorously and completely in [32], where the general case of a spin $\frac{1}{2}$-loop coupled to background scalar, pseudoscalar, vector, axial-vector, and antisymmetric tensor fields was treated. However in both cases the construction of the path integrals had to be done separately for the real and the imaginary parts of the (Euclidean) effective action. Moreover, for the imaginary part certain "insertion operators" had to be defined in addition to the worldline lagrangean. In part I of the present series [33] the case of a vector–axialvector background was considered, and a path integral representation for the corresponding one-loop effective action derived which avoids both of these complications. Let us repeat this derivation shortly.

One-loop axialvector and vector current correlators can be extracted from the determinant of the non-Hermitian operator $H - im$ where

$$H = \not p + \not A + \gamma_5 \not A_5\,, \qquad (p = -i\partial) \tag{1.1}$$

(in Euclidean space) [4, 34]. It has proved possible to deal with this functional determinant without resorting to a decomposition into its real and imaginary parts [4, 34]. The basic identity used in our present approach is [35]

$$\begin{aligned}(\not p + \not A + \gamma_5 \not A_5)^2 = {} & (p_\mu + A_\mu - \gamma_5 \sigma_{\mu\nu} A_5^\nu)^2 + (D-2)A_5^2 + \\ & + iA^\nu_{5,\nu}\gamma_5 - \frac{i}{2}\sigma_{\mu\nu}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\,,\end{aligned} \tag{1.2}$$

valid for any even dimension D in Euclidean spacetime.[2] Our object of interest is the heat kernel

$$K(y,z;T) = \langle y | e^{-H^2 T} | z \rangle\,. \tag{1.3}$$

[1] In the nonabelian case the arguments of [13] fall somewhat short of a complete rederivation of the Bern-Kosower rules.

[2] We work in the Euclidean throughout with a positive definite metric $g_{\mu\nu} = \mathrm{diag}(+ + s+)$. Our Dirac matrix conventions are (with some abuse of notation) $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, $\gamma_5^2 = \mathbb{1}$, $\gamma_{\mu,5}^\dagger = \gamma_{\mu,5}$,

Standard arguments [36] show that we can represent K as a path-ordered quantum mechanical path integral as follows,

$$K(y,z;T) = \int_{x(0)=z}^{x(T)=y} \mathcal{D}x(\tau)\mathcal{P}\exp\left[-\int_0^T \mathcal{L}(x,\gamma)\,d\tau\right], \tag{1.4}$$

where

$$\begin{aligned}\mathcal{L}(x,\gamma) = &\frac{\dot{x}^2}{4} + iA\dot{x} - \frac{i}{4}[\gamma^\mu,\gamma^\nu](\partial^\mu A^\nu - \partial^\nu A^\mu) - \\ &-\frac{i}{2}\gamma_5[\gamma^\mu,\gamma^\nu]A_5^\nu\dot{x}^\mu + i\gamma_5\partial_\mu A_5^\mu + (D-2)A_5^2\,.\end{aligned} \tag{1.5}$$

In [33] only the effective action was considered, which is obtained from the heat-kernel K by taking the functional trace and integrating over proper-time. The coherent state method [32, 37] was then used to transform the trace over spinor indices into a Grassmann path integral, with the result

$$\Gamma[A,A_5] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\int\mathcal{D}x\int\mathcal{D}\psi\;\mathrm{e}^{-\int_0^T d\tau\, L(\tau)} \tag{1.6}$$

$$\begin{aligned}L = &\frac{1}{4}\dot{x}^2 + \frac{1}{2}\psi\dot{\psi} + i\dot{x}A - i\psi F\psi + \\ &+ i\hat{\gamma}_5\Big[-2\dot{x}\psi\psi A_5 + \partial A_5\Big] + (D-2)A_5^2\,.\end{aligned} \tag{1.7}$$

Although our use of the coherent state method completely parallels [32] we spell out the details of this transformation in appendix A for completeness (for $D=4$). The result is a formal correspondence $[\gamma^\mu,\gamma^\nu]\to 4\psi^\mu\psi^\nu$, $\gamma_5\to\hat{\gamma}_5$. This formula for the effective action is valid for any even spacetime dimension, and also for the massive case. For fixed T one has to calculate a coordinate path integral $\int\mathcal{D}x$ over the space of all closed loops in spacetime with periodicity T, and a Grassmann path integral $\int\mathcal{D}\psi$. The boundary conditions on the Grassmann path integral are, after expansion of the interaction exponential, determined by the power of $\hat{\gamma}_5$ appearing in a given term; they are (anti) periodic with period T if that power is even (odd). After the boundary conditions are determined $\hat{\gamma}_5$ can be replaced by unity.

In part I it was shown how to apply this path integral to a calculation of the effective action itself in the heat kernel expansion, both for its anomalous and non-anomalous parts. In the present paper we exploit it for deriving master formulas for the one-loop (off-shell) amplitudes with an arbitrary number of vector and axial-vector legs. Those are extensions of the string-inspired master formula for QED photon scattering, which in turn is a special case of the QCD Bern-Kosower master formula [10].

$\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu,\gamma_\nu]$, $\mu,\nu = 1,\ldots,D$. The Euclidean ε - tensor is defined by $\varepsilon_{1sD} = +1$. We do not differentiate between superscripts and subscripts. The corresponding Minkowski space formulas are obtained by $g_{\mu\nu}\to\eta_{\mu\nu} = \mathrm{d}iag(-+s+)$,$k_D\to -ik^0$,$T\to is$,$\varepsilon^{\mu_1\cdots\mu_D}\to i\varepsilon^{\mu_1\cdots\mu_D}$,$\varepsilon^{0\ldots(D-1)} = +1$.

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We then investigate alternative methods for computing the general element of the heat kernel. Although these methods apply to both diagonal and off-diagonal calculations, for the latter we must restrict our attention to the massless fermion since a mass parameter turns out to be problematic. We use the Fradkin-Gitman (FG) formalism [38] to eliminate the Dirac matrices in (1.4) in favour of Grassmann parameters; for this we recall the approach of [39]. We then demonstrate how both $\mathcal{P}\exp\int_0^T d\tau\, A^{\mu\nu}(\tau)\sigma^{\mu\nu}$ and $\mathcal{P}\exp\int_0^T d\tau\, \vec{A}(\tau)\vec{\sigma}$ can be written as closed linear combination of spin matrices, thereby completely circumventing the need for introducing any Grassmann integration. The latter path-ordered operator is relevant in connection with our observation that the matrix valued lagrangean appearing in (1.4) commutes with the chirality operator γ_5. We exploit this property by showing how the decomposition of the SO(4) generators $\sigma_{\mu\nu}$ appearing in (1.4) into the generators of the SU(2) subgroups by use of the 't Hooft symbols $\eta^{\pm}_{a\mu\nu}$ (originally introduced in the context of instantons [40]) can simplify the functional integral considered in (1.4). The utility of this approach is illustrated by a calculation of the axial anomaly.

We also demonstrate that when the Dirac structure of K is decomposed using the FG approach, the resulting zero-mode dependence leads to a very natural functional interpretation of the "Feynman rules" occurring in the coherent state method. We show how external fermions can be integrated into the FG approach and how this leads to a suitable unification of the background axial, vector and spinor cases. The possibility of handling isospin in the same way that we have treated ordinary spin is also considered briefly. For the pure vector case we also examine the interplay between worldline supersymmetry and the fermionic boundary conditions, using a theorem from $0+1$ dimensional topological field theory.

2. A master formula for the one-loop vector–axialvector amplitudes

Generally, as a first step in any evaluation of the double path integral in (1.6) one has to eliminate the zero mode(s) contained in it. For $\int\mathcal{D}x$ this is done by fixing the average position of the loop, i.e. one writes

$$\begin{aligned} x^\mu(\tau) &= x_0^\mu + y^\mu(\tau) \\ \int \mathcal{D}x &= \int dx_0 \int \mathcal{D}y\,, \end{aligned} \tag{2.1}$$

where

$$x_0^\mu \equiv \frac{1}{T}\int_0^T d\tau\, x^\mu(\tau)\,. \tag{2.2}$$

The Grassmann path integral has a zero-mode only in the periodic case. This must again be separated out:

$$\psi^\mu(\tau) = \psi_0^\mu + \xi^\mu(\tau)\,, \qquad \int_0^T d\tau\, \xi^\mu(\tau) = 0\,. \tag{2.3}$$

The zero mode integration then produces the ε-tensor expected for a spinor loop with an odd number of axial insertions

$$\int d^D\psi_0 \psi_0^{\mu_1}\psi_0^{\mu_2} s \psi_0^{\mu_D} = \varepsilon^{\mu_1 s \mu_D} \,. \tag{2.4}$$

The remaining y- and ξ-path integrals have invertible kinetic terms. In the "string-inspired" formalism they are performed using worldline Green's functions adapted to the boundary conditions

$$\begin{aligned}
\langle y^\mu(\tau_1)\, y^\nu(\tau_2)\rangle &= -g^{\mu\nu} G_B(\tau_1,\tau_2)\,,\\
\langle \psi^\mu(\tau_1)\, \psi^\nu(\tau_2)\rangle_A &= g^{\mu\nu}\frac{1}{2} G_F(\tau_1,\tau_2)\,,\\
\langle \xi^\mu(\tau_1)\, \xi^\nu(\tau_2)\rangle_P &= g^{\mu\nu}\frac{1}{2} \dot G_B(\tau_1,\tau_2)\,,
\end{aligned} \tag{2.5}$$

where[3]

$$\begin{aligned}
G_B(\tau_1,\tau_2) &= \mid \tau_1-\tau_2 \mid - \frac{(\tau_1-\tau_2)^2}{T}\,,\\
\dot G_B(\tau_1,\tau_2) &= \mathrm{sign}(\tau_1-\tau_2) - 2\frac{\tau_1-\tau_2}{T}\,,\\
G_F(\tau_1,\tau_2) &= \mathrm{sign}(\tau_1-\tau_2)\,.
\end{aligned} \tag{2.6}$$

A "dot" always refers to a derivative in the first variable. The last information needed are the free Gaussian path integral determinants, which with the present conventions are[4]

$$\begin{aligned}
\int \mathcal{D}y\, \mathrm{e}^{-\int_0^T d\tau \frac{1}{4}\dot y^2} &= (4\pi T)^{-\frac{D}{2}}\,,\\
\int_A \mathcal{D}\psi\, \mathrm{e}^{-\int_0^T d\tau \frac{1}{2}\psi\dot\psi} &= 2^{\frac{D}{2}} =: N_A\,,\\
\int_P \mathcal{D}\xi\, \mathrm{e}^{-\int_0^T d\tau \frac{1}{2}\xi\dot\xi} &= (-1)^{\frac{D}{2}} =: N_P\,.
\end{aligned} \tag{2.7}$$

To extract the scattering amplitude from the effective action, as usual we must specialize the background fields to plane waves, and then keep the part of the effective action which is linear in all polarization vectors. As a preliminary step, it is convenient to linearize the term quadratic in A_5 by introducing an auxiliary path integration, writing

$$\exp\left[-(D-2)\int_0^T d\tau A_5^2\right] = \int \mathcal{D}z \exp\left[-\int_0^T d\tau \left(\frac{z^2}{4} + i\sqrt{D-2}\,zA_5\right)\right]. \tag{2.8}$$

[3]There is a certain freedom in the choice of the bosonic correlator. Our G_B is the one generally used in the "string-inspired" formalism. It yields the same final results as the one employed in [39], as discussed in [30].

[4]When comparing to [33] note that we have shifted a factor of $2^{\frac{D}{2}}$ from the overall factor to the fermionic path integral.

This allows us to define an axial-vector vertex operator as follows

$$V_{A_5}[k,\varepsilon] \equiv \int_0^T d\tau \Big(i\varepsilon k + 2\varepsilon\psi\dot{x}\psi + \sqrt{D-2}\,\varepsilon z \Big)\, \mathrm{e}^{ikx}\,. \tag{2.9}$$

Before using this vertex operator for Wick-contractions, as usual it is convenient to formally rewrite it as a linearized exponential. This can be done introducing Grassmann variables

$$\begin{aligned} V_{A_5}[k,\varepsilon] = \int_0^T d\tau \int d\theta \exp\Big[& ikx + i\theta\varepsilon k + \sqrt{2}\varepsilon\psi + \sqrt{2}\theta\dot{x}\psi + \\ & + \sqrt{D-2}\,\theta\varepsilon z\Big]\Big|_{\mathrm{lin}(\varepsilon)}\,. \end{aligned} \tag{2.10}$$

Here θ is a Grassmann variable with $\int d\theta\theta = 1$, and ε must now also be formally treated as Grassmannian. The vectors are represented by the usual photon vertex operator familiar from string theory

$$\begin{aligned} V_A[k,\varepsilon] &\equiv \int_0^T d\tau \Big[\varepsilon\dot{x} + 2i\varepsilon\psi k\psi\Big]\, \mathrm{e}^{ikx} \\ &= \int_0^T d\tau \int d\theta \exp\Big[ik(x+\sqrt{2}\theta\psi) + \varepsilon(-\theta\dot{x} + \sqrt{2}\psi)\Big]\Big|_{\mathrm{lin}(\varepsilon)}\,, \end{aligned} \tag{2.11}$$

which we have also rewritten as a linearized exponential. Those definitions allow us to represent the one-loop amplitude with M vectors and N axialvectors in the following way:

$$\begin{aligned} \Gamma[\{k_i,\varepsilon_i\},\{k_{5j},\varepsilon_{5j}\}] = & -\frac{1}{2}N_{A,P}(-i)^{M+N}\int_0^\infty \frac{dT}{T}\, e^{-m^2T}(4\pi T)^{-\frac{D}{2}} \times \\ & \times\Big\langle V_A[k_1,\varepsilon_1] s V_A[k_M,\varepsilon_M] V_{A_5}[k_{51},\varepsilon_{51}] s V_{A_5}[k_{5N},\varepsilon_{5N}]\Big\rangle \end{aligned} \tag{2.12}$$

where the global sign refers to the ordering $\varepsilon_1\varepsilon_2\ldots\varepsilon_M\varepsilon_{51}\varepsilon_{52}\ldots\varepsilon_{5N}$ of the polarisation vectors. It is then straightforward to perform the bosonic path integrations

$$\begin{aligned} &\int \mathcal{D}x \int \mathcal{D}z\, V_A[k_1,\varepsilon_1]\ldots V_{A_5}[k_{5N},\varepsilon_{5N}]\, \mathrm{e}^{-\int_0^T d\tau\left(\frac{\dot{x}^2}{4}+\frac{z^2}{4}\right)} = \\ &= (4\pi T)^{-\frac{D}{2}} \int_0^T d\tau_1 s \int d\theta_M \int_0^T d\tau_{51} s \int d\theta_{5N} \times \\ &\times \exp\Big\{\frac{1}{2}G_{BIJ}K_IK_J - i\theta_i\dot{G}_{BiJ}\varepsilon_iK_J - i\sqrt{2}\theta_{5i}\dot{G}_{BiJ}\psi_iK_J - \\ &\qquad -\frac{1}{2}\ddot{G}_{Bij}\theta_i\theta_j\varepsilon_i\varepsilon_j - \sqrt{2}\ddot{G}_{Bij}\theta_i\theta_{5j}\varepsilon_i\psi_j - \ddot{G}_{Bij}\theta_{5i}\theta_{5j}\psi_i\psi_j + \\ &\qquad + \sqrt{2}(\varepsilon_i\psi_i + \varepsilon_{5j}\psi_j) + \sqrt{2}i\theta_ik_i\psi_i + i\theta_{5i}\varepsilon_{5i}k_{5i} - \\ &\qquad - (D-2)\delta(\tau_i-\tau_j)\theta_{5i}\theta_{5j}\varepsilon_{5i}\varepsilon_{5j}\Big\}\Big|_{\mathrm{lin}(\{\varepsilon_i\};\{\varepsilon_{5j}\})}\,. \end{aligned} \tag{2.13}$$

Here and in the following all small repeated indices run over either the vector or the axial vector indices, while capital repeated indices run over all of them ($\{K_I\}$ denotes the set of all external momenta). The remaining ψ-path integral is still Gaussian. For its performance we must now distinguish between even and odd numbers of axialvectors. For the antiperiodic case, N even, there is no zero-mode, and the integration can still be done in closed form. The only complication is the existence of the term $-\ddot{G}_{Bij}\theta_{5i}\theta_{5j}\psi_i\psi_j$. It modifies the worldline propagator G_F to

$$G^{(N)}_{F12} \equiv 2\langle\tau_1 \mid \Bigl(\partial + 2B^{(N)}\Bigr)^{-1} \mid \tau_2\rangle\,, \tag{2.14}$$

where $B^{(N)}$ denotes the operator with integral kernel

$$B^{(N)}(\tau_1,\tau_2) = \delta(\tau_1-\tau_i)\theta_{5i}\ddot{G}_{Bij}\theta_{5j}\delta(\tau_j-\tau_2) \tag{2.15}$$

($B^{(N)}$ acts trivially on the Lorentz indices, which we suppress in the following). Expanding the right hand side of eq. (2.14) in a geometric series and resumming one obtains a matrix representation for $G^{(N)}_{F12}$

$$G^{(N)}_F = \frac{G_F}{\mathbb{1}+\ddot{\Theta}G_F} = G_F - G_F\ddot{\Theta}G_F + s\,. \tag{2.16}$$

Here $\ddot{\Theta}_{ij}$ is the antisymmetric $N\times N$ matrix with entries $\theta_{5i}\ddot{G}_{Bij}\theta_{5j}$ (no summation). Moreover, the fermionic path integral determinant changes by a factor

$$\mathrm{Det}^{\frac{1}{2}}\Bigl(\mathbb{1}+2B^{(N)}\partial^{-1}\Bigr) = \det(\mathbb{1}+\ddot{\Theta}G_F)^{\frac{D}{2}} \tag{2.17}$$

as is easily seen using the $\ln\det = \mathrm{tr}\ln$ - formula (note that on the left hand side we have a functional determinant, on the right hand side the determinant of a $N\times N$ matrix). Using these results the fermionic path integral can be eliminated, yielding the following master formula for this amplitude

$$\begin{aligned}
\Gamma_{\rm even}[\{k_i,\varepsilon_i\};\{k_{5j},\varepsilon_{5j}\}] = & -\frac{N_A}{2}(-i)^{M+N}\int_0^\infty\frac{dT}{T}\,e^{-m^2T}(4\pi T)^{-\frac{D}{2}}\times\\
& \times\int_0^T d\tau_1\int d\theta_1 s\int_0^T d\tau_{5N}\int d\theta_{5N}\,\det(\mathbb{1}+\ddot{\Theta}G_F)^{\frac{D}{2}}\times\\
& \times\exp\Bigl\{\frac{1}{2}G_{BIJ}K_IK_J - i\theta_i\dot{G}_{BiJ}\varepsilon_iK_J - \frac{1}{2}\ddot{G}_{Bij}\theta_i\theta_j\varepsilon_i\varepsilon_j -\\
& \quad -\frac{G^{(N)}_{Fij}}{2}\Bigl(\varepsilon_i + i\theta_ik_i + \varepsilon_{5i} - i\theta_{5i}\dot{G}_{BiR}K_R + \ddot{G}_{Bir}\theta_{5i}\theta_r\varepsilon_r\Bigr)\times\\
& \quad\times\Bigl(\varepsilon_j + i\theta_jk_j + \varepsilon_{5j} - i\theta_{5j}\dot{G}_{BjS}K_S + \ddot{G}_{Bjs}\theta_{5j}\theta_s\varepsilon_s\Bigr) + i\theta_{5i}\varepsilon_{5i}k_{5i} -\\
& \quad -(D-2)\delta(\tau_i-\tau_j)\theta_{5i}\theta_{5j}\varepsilon_{5i}\varepsilon_{5j}\Bigr\}\Big|_{\mathrm{lin}(\{\varepsilon_i\};\{\varepsilon_{5j}\})}\,.
\end{aligned} \tag{2.18}$$

As is usual in the string-inspired formalism all momenta are *outgoing*. Let us verify the correctness of this formula for the case of the massive 2-point axialvector function in four dimensions. Expanding out the exponential as well as the determinant factor, and performing the two θ-integrals, we obtain the following parameter integral

$$\begin{aligned}\Gamma[k_1,\varepsilon_1;k_2,\varepsilon_2] = 2\int_0^\infty \frac{dT}{T}\, e^{-m^2T}(4\pi T)^{-\frac{D}{2}}\int_0^T d\tau_1\int_0^T d\tau_2 \\ \times \mathrm{e}^{G_{B12}k_1k_2}\Big\{2(D-2)\delta(\tau_1-\tau_2)\varepsilon_1\varepsilon_2 - (D-1)\ddot{G}_{B12}G_{F12}^2\varepsilon_1\varepsilon_2 \\ - G_{F12}^2\dot{G}_{B12}^2(\varepsilon_1\varepsilon_2k_1k_2 - \varepsilon_1k_1\varepsilon_2k_2) - \varepsilon_1k_1\varepsilon_2k_2\Big\}\,.\end{aligned} \tag{2.19}$$

As usual in this type of calculation we rescale $\tau_{1,2} = Tu_{1,2}$, and use the translation invariance in τ to set $u_2 = 0$. Setting also $k = k_1 = -k_2$ this leads to

$$\begin{aligned}\Gamma^{\mu\nu}[k] = 2\int_0^\infty \frac{dT}{T}\, e^{-m^2T}(4\pi T)^{-\frac{D}{2}}\Big\{2(D-2)Tg^{\mu\nu} - 2(D-1)Tg^{\mu\nu} + \\ + \int_0^1 du\, \mathrm{e}^{-Tu(1-u)k^2}\Big[2(D-1)Tg^{\mu\nu} + (1-2u)^2T^2(g^{\mu\nu}k^2 - k^\mu k^\nu) + T^2k^\mu k^\nu\Big]\Big\}\end{aligned} \tag{2.20}$$

In the massless case the first two terms in braces do not contribute in dimensional regularization, since they are of tadpole type. For the remaining terms both integrations are elementary, and the result is, using Γ-function identities, easily identified with the standard result for the massless QED vacuum polarisation. A suitable integration by part verifies the agreement with field theory also for the massive case. Here the tadpole terms do contribute, and the comparison shows that to get the precise D-dependence of the amplitude, appropriate to dimensional regularisation using an anticommuting γ_5, it was essential to keep the explicit D-dependence of the A_5^2-term in the worldline lagrangean.

For an odd number of axial vectors, we need to go back to eq. (2.13) and replace ψ by $\psi_0 + \xi$. The $\mathcal{D}\xi$-path integral is then executed in the same way as before, but with the propagator G_F changed to $\dot{G}_B$. The final result becomes

$$\begin{aligned}\Gamma_{\text{odd}}[\{k_i,\varepsilon_i\};\{k_{5j},\varepsilon_{5j}\}] = -\frac{N_P}{2}(-i)^{M+N}\int_0^\infty \frac{dT}{T}\, e^{-m^2T}(4\pi T)^{-\frac{D}{2}} \times \\ \times \int_0^T d\tau_1 \int d\theta_1 s\int_0^T d\tau_{5N}\int d\theta_{5N}\,\det(\mathbb{1} + \ddot{\Theta}\dot{G}_B)^{\frac{D}{2}}\int d^D\psi_0 \times \\ \times \exp\Big\{\frac{1}{2}G_{BIJ}K_IK_J - i\theta_i\dot{G}_{BiJ}\varepsilon_iK_J - \frac{1}{2}\ddot{G}_{Bij}\theta_i\theta_j\varepsilon_i\varepsilon_j - \\ -\frac{\dot{G}_{Bij}^{(N)}}{2}\Big(\varepsilon_i + i\theta_ik_i + \varepsilon_{5i} - i\theta_{5i}\dot{G}_{BiR}K_R + \ddot{G}_{Bir}\theta_{5i}\theta_r\varepsilon_r + \sqrt{2}\ddot{G}_{Bir}\theta_{5i}\theta_{5r}\psi_0\Big) \times \\ \times\Big(\varepsilon_j + i\theta_jk_j + \varepsilon_{5j} - i\theta_{5j}\dot{G}_{BjS}K_S + \ddot{G}_{Bjs}\theta_{5j}\theta_s\varepsilon_s + \sqrt{2}\ddot{G}_{Bjs}\theta_{5j}\theta_{5s}\psi_0\Big) + \\ + i\theta_{5i}\varepsilon_{5i}k_{5i} - (D-2)\delta(\tau_i-\tau_j)\theta_{5i}\theta_{5j}\varepsilon_{5i}\varepsilon_{5j} - i\sqrt{2}\theta_{5i}\dot{G}_{BiJ}\psi_0K_J - \\ -\sqrt{2}\ddot{G}_{Bij}\theta_i\theta_{5j}\varepsilon_i\psi_0 + \sqrt{2}(\sum\varepsilon_i + \sum\varepsilon_{5j})\psi_0 + \sqrt{2}i\theta_ik_i\psi_0\Big\}\Big|_{\text{lin}(\{\varepsilon_i\};\{\varepsilon_{5j}\})}\end{aligned} \tag{2.21}$$

Here $\dot{G}_B^{(N)}$ is defined analogously to eq. (2.16),

$$\dot{G}_B^{(N)} = \frac{\dot{G}_B}{\mathbb{1} + \ddot{\Theta}\dot{G}_B} \,. \tag{2.22}$$

The integrand still depends on the zero-mode ψ_0, which is to be integrated according to eq. (2.4). For a given dimension of space-time this can, of course, be easily worked out explicitly.

3. The Dirac algebra using FG and deducing spin factors

Of course it is desirable to have a representation for the heat-kernel which applies to both diagonal and off-diagonal elements. The presence of a non-vanishing fermion mass in the latter case becomes an essential complication which we will not consider in the present paper. Moreover we specialize to $D = 4$, since we are now going to use more specific properties of the Dirac algebra, although many of our methods have analogues in other dimensions. For this generalization the coherent state method seems less convenient, and we rather use the "Weyl symbol" formalism [41, 38, 39], as follows. We begin by presenting two equations of which we will make frequent use [38, 42, 2, 43]:

$$\mathcal{P}\exp\int_0^T d\tau f\left(\tau,\gamma\right) = \hat{\mathcal{P}}\exp\int_0^T d\tau f\left(\tau,\tfrac{\partial}{\partial\rho}\right)\,\mathcal{P}\exp\int_0^T d\tau\rho(\tau)\gamma\bigg|_{\rho=0} \tag{3.1}$$

$$\begin{aligned}\mathcal{P}\exp\int_0^T d\tau\rho(\tau)\gamma = \exp\left[-\frac{1}{2}\int_0^T \mathrm{sign}(\tau_1-\tau_2)\rho(\tau_1)\rho(\tau_2)\,d\tau_1 d\tau_2\right]\times\\ \times\exp\int_0^T d\tau\rho(\tau)\gamma\,,\end{aligned} \tag{3.2}$$

where $\rho^\mu(\tau)$, with $\mu = 1,\ldots,4$, are Grassmann variables which anticommute[5] with γ^ν, and $f(\tau,\gamma)$ is a linear combination of antisymmetrized products of γ-matrices. Path ordering of matrices (both γ and later $\vec{\sigma}$), is denoted by $\mathcal{P}$ while $\hat{\mathcal{P}}$ denotes path ordering of Grassmann derivatives $\frac{\partial}{\partial\rho}$.

Equations (2.6), (3.1) and (3.2) can be used to convert (1.4) to

$$\begin{aligned}K(y,z;T) = \int_{x(0)=z}^{x(T)=y}\mathcal{D}x(\tau)\hat{\mathcal{P}}\exp\int_0^T d\tau\mathcal{L}\left(x,\tfrac{\partial}{\partial\rho}\right)\times\\ \times\exp\left[-\frac{1}{2}\int_0^T d\tau_1 d\tau_2\, G_F(\tau_1,\tau_2)\rho(\tau_1)\rho(\tau_2)\right]\exp\int_0^T d\tau\rho(\tau)\gamma\bigg|_{\rho=0}\,.\end{aligned} \tag{3.3}$$

[5]The anticommutation of γ with ρ (and θ to be introduced later) is a matter of algebraic convenience. It should be possible to formulate the problem with the assumption that γ is a c-number valued matrix, in which case $\hat{\mathcal{P}}$ would have to denote *reversed* path-ordering. However, we follow [38] for convenience and retain the convention that γ is a Grassmann quantity.

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In this equation, the $\hat{\mathcal{P}}$ operator above is actually superfluous since the lagrangean $\mathcal{L}$ is even in the anticommuting variables.

Notice that the last exponential in equations (3.2) and (3.3) depends only on the finite set of so-called *zero modes* of the Grassmann field, $T\rho_0^\mu = \int_0^T d\tau\rho^\mu(\tau)$, so the Taylor series truncates.

Equation (3.2) is a special case of Wick's theorem which relates the $\mathcal{P}$ and *Sym* orderings of the operator $\exp\int_0^T d\tau\rho\gamma$. A more extensive discussion can be found in [42]. The relation follows from generalized form of Wick's theorem (see, for instance, page 60 of [44]) which can be used to relate any two operator orderings of a given functional. It's appearance in (3.3) allows us to draw strong connections with the Feynman rules used in the string-inspired formalism, enumerated in section 2. For this we defer the reader to appendix B.

One method of dealing with (1.4) is to use the Fradkin-Gitman approach [38] to replace the time ordered exponential involving Dirac matrices by an integral over Grassmann variables. The result is

$$K(y,z;T) \propto \exp\left(i\gamma\frac{\partial}{\partial\theta}\right)\int_{x(0)=z}^{x(T)=y}\mathcal{D}x\int_{\psi(0)+\psi(T)=\sqrt{2}\theta}\mathcal{D}\psi\, e^{\frac{1}{2}\psi(T)\psi(0)+\int_0^T d\tau L(\tau)}\Big|_{\theta=0}, \tag{3.4}$$

where $L(\tau)$ is defined in (1.7). This approach has been exploited in [39].

However, it is a bit awkward to work with (3.4), due to having the quantity θ arising in the boundary condition on the Grassmann parameter ψ over which we are integrating. We can simplify (1.4) directly by essentially generalizing a well-known functional identity [2]. We start by noting the standard result

$$\int\mathcal{D}\omega_i \exp\left[\tfrac{1}{2}\omega_i A_{ij}\omega_j + \rho_i\omega_i\right] = \det{}^{1/2}A\exp\tfrac{1}{2}\rho_i A_{ij}^{-1}\rho_j\,. \tag{3.5}$$

where ω_i and ρ_i are Grassmann and $A_{ij} = -A_{ji}$. From (3.5), it follows that

$$\begin{aligned} Q \equiv\; & \exp\left[\frac{1}{2}\frac{\partial}{\partial\rho_i}B_{ij}\frac{\partial}{\partial\rho_j}\right]\exp\left[\tfrac{1}{2}\rho_i A_{ij}^{-1}\rho_j + \rho_i c_i\right]\Big|_{\rho=0} \\ =\; & \exp\left[\frac{1}{2}\frac{\partial}{\partial\rho_i}B_{ij}\frac{\partial}{\partial\rho_j}\right]\det{}^{-1/2}A\;\times \\ & \times\int\mathcal{D}\omega_i\exp\left[\tfrac{1}{2}\omega_i A_{ij}\omega_j + \rho_i(\omega_i + c_i)\right]\Big|_{\rho=0} \end{aligned} \tag{3.6}$$

(where c_i is Grassmann and $B_{ij} = -B_{ji}$), so that

$$Q = \det{}^{1/2}A^{-1}\int\mathcal{D}\omega_i\exp\left[\tfrac{1}{2}\omega_i(A_{ij}+B_{ij})\omega_j + c_i B_{ij}\omega_j + \tfrac{1}{2}c_i B_{ij}c_j\right]. \tag{3.7}$$

Again, employing (3.5) to compute the functional integral in (3.7), we obtain,

$$\begin{aligned} Q &= \det{}^{1/2}\left(\mathbb{1} + A^{-1}B\right)\exp\left[-\tfrac{1}{2}c_i B_{ij}\left(1 + A^{-1}B\right)_{jk}^{-1}\left(A^{-1}B\right)_{kl}c_l + \tfrac{1}{2}c_i B_{ij}c_j\right] && (3.8)\\ &= \det{}^{1/2}\left(\mathbb{1} + A^{-1}B\right)\exp\left[\tfrac{1}{2}c_i\left(A(A+B)^{-1}B\right)_{ij}c_j\right]. && (3.9) \end{aligned}$$

Although equation (3.9) is formally simpler, equation (3.8) turns out to be more convenient for our (perturbative) calculations below. Remembering that c_i in (3.8) is a Grassmann variable, we see that the second exponential in (3.8) can be expanded in a finite series provide there are a finite number of c_i's.

The result (3.8) is useful for treating the dependence of the path integral in equation (1.4) on the gamma matrices. To see this, we now consider the functional

$$I_4[B_4] = \mathcal{P}\exp\frac{1}{2}\int_0^T d\tau B_4^{\mu\nu}(\tau)\gamma^\mu\gamma^\nu\,, \tag{3.10}$$

which is relevant for the case at hand (1.4) if the antisymmetric tensor $B_4(\tau)$ is given by

$$B_4^{\mu\nu} = -\tfrac{i}{2}F^{\mu\nu} + \tfrac{i}{2}\epsilon^{\mu\nu\lambda\sigma}\dot{x}^\lambda A_5^\sigma\,.$$

We write, using (3.1) and (3.2),

$$\begin{aligned} I_4[B_4] = \exp\int_0^T d\tau\frac{1}{2}\left[B_4^{\mu\nu}(\tau)\frac{\partial}{\partial\rho^\mu}\frac{\partial}{\partial\rho^\nu}\right]\times \\ \times\exp\left[-\frac{1}{2}\int_0^T d\tau_1 d\tau_2 G_F(\tau_1,\tau_2)\rho(\tau_1)\rho(\tau_2)\right]\exp\left[\int_0^T d\tau\rho(\tau)\gamma\right]\Bigg|_{\rho=0} \end{aligned} \tag{3.11}$$

which becomes

$$\begin{aligned} I_4[B_4] = \exp\int_0^T d\tau\frac{1}{2}\left[B_4^{\mu\nu}(\tau)\frac{\partial}{\partial\rho^\mu}\frac{\partial}{\partial\rho^\nu}\right]\times \\ \times\exp\left[-\frac{1}{2}\int_0^T d\tau_1 d\tau_2 G_F(\tau_1,\tau_2)\rho(\tau_1)\rho(\tau_2)\right]\times \\ \times\exp\left[\gamma\frac{\partial}{\partial\theta}\right]\exp\left[\int_0^T d\tau\rho(\tau)\theta\right]\Bigg|_{\theta=\rho=0}\,, \end{aligned} \tag{3.12}$$

where θ^μ is another Grassmann variable that anticommutes with both $\rho^\mu(\tau)$ and γ^μ. Under the appropriate identifications, we see that (3.8) can be used to reduce (3.12) to

$$\begin{aligned} \mathcal{P}\exp\frac{1}{2}\int_0^T d\tau B_4^{\mu\nu}(\tau)\gamma^\mu\gamma^\nu = \overset{1/2}{\det}(\mathbb{1}-G_F B_4)\exp\left[\gamma\frac{\partial}{\partial\theta}\right]\times \\ \times\exp\int_0^T d\tau_1 d\tau_2 d\tau_3\left[\frac{1}{2}\theta_\mu B_4^{\mu\nu}(\tau_1)\left\langle\tau_1\left|(\mathbb{1}-G_F B_4)^{-1}\right|\tau_2\right\rangle_{\nu\lambda}G_F(\tau_2,\tau_3)B_4^{\lambda\sigma}(\tau_3)\theta_\sigma\right]\times \\ \times\exp\left[\int_0^T d\tau\frac{1}{2}\theta_\mu B_4^{\mu\nu}(\tau)\theta_\nu\right]\Bigg|_{\theta=0}\,, \end{aligned} \tag{3.13}$$

where (using a notation slightly different from that of sect. 2) $G_F B_4$ is the Lorentz matrix-valued integral operator with kernel $\langle\tau_1|G_F B_4|\tau_2\rangle^{\mu\nu} = G_F(\tau_1,\tau_2)B_4^{\mu\nu}(\tau_2)$. If we now expand $e^{\gamma\frac{\partial}{\partial\theta}}$, only the terms up to order $\left(\frac{\partial}{\partial\theta}\right)^4$ contribute due to the Grassmann nature of the θ. We consequently find that

$$I_4 = \overset{1/2}{\det}(\mathbb{1}-G_F B_4)\left[1+\sigma_{\mu\nu}X^{\mu\nu}+\frac{1}{2!}\varepsilon_{\mu\nu\lambda\sigma}X^{\mu\nu}X^{\lambda\sigma}\gamma_5\right],$$

where

$$X^{\mu\nu} = \frac{1}{2}\int_0^T d\tau_2 d\tau_2 d\tau_3 \left[B_4^{\mu\lambda}(\tau_1)\left\langle\tau_1\left|(\mathbb{1}-G_F B_4)^{-1}\right|\tau_2\right\rangle_{\lambda\sigma} G_F(\tau_2,\tau_3)B_4^{\sigma\nu}(\tau_3)\right] + \\ + \int_0^T d\tau B_4^{\mu\nu}(\tau)\,. \tag{3.14}$$

The same procedure can be used to derive a similar decomposition when Pauli spin matrices $\vec{\sigma}$ rather than Dirac matrices γ_μ occur. (This will prove useful in connection with our SO(4) = SU(2) × SU(2) decomposition (3.22) below.)

If we consider

$$I_3[\vec{A}] = \mathcal{P}\exp\int_0^T d\tau\vec{A}(\tau)\vec{\sigma} \tag{3.15}$$

we note that $2i\sigma_a = \varepsilon_{abc}\sigma_b\sigma_c$, allowing us to write

$$I_3 = \mathcal{P}\exp\frac{1}{2}\int_0^T d\tau B_3^{ab}(\tau)\sigma_a\sigma_b\,, \tag{3.16}$$

where $iB_3^{ab}(\tau) \equiv \varepsilon^{abc}A_c$. The analogue of (3.14) is then (remembering that $\vec{\sigma}$ is only a triplet)

$$I_3 = \overset{1/2}{\det}\left(\mathbb{1}-G_F B_3\right)(1+Y)\,, \tag{3.17}$$

where

$$Y = \frac{1}{2}[\sigma_a,\sigma_b]\left\{\frac{1}{2}\int_0^T d\tau_1 d\tau_2 d\tau_3\left[B_3^{ac}(\tau_1)\left\langle\tau_1\left|(\mathbb{1}-G_F B_3)^{-1}\right|\tau_2\right\rangle_{cd} G_F(\tau_2,\tau_3)B_3^{db}(\tau_3)\right]\right. \\ \left. + \frac{1}{2}\int_0^T d\tau B_3^{ab}(\tau)\right\}. \tag{3.18}$$

This is a generalization of the spin factor of Polyakov [45].

There is an alternate expression for I_3 in (3.15) that also proves useful below. In this approach we apply the analogue of equations (3.1) and (3.2) to arrive at

$$\begin{aligned} I_3[\vec{A}] &= \mathcal{P}\exp\int_0^T \vec{A}(\tau)\vec{\sigma} \\ &= \hat{\mathcal{P}}\exp\left[\int_0^T d\tau\vec{A}(\tau)\frac{\partial}{\partial\vec{\rho}(\tau)}\right]\mathcal{P}\exp\left[\int_0^T d\tau\vec{\rho}(\tau)\vec{\sigma}\right]\Bigg|_{\vec{\rho}=0}\,, \end{aligned} \tag{3.19}$$

where again, $\vec{\rho}(\tau)$ is a Grassmann source which anticommutes with the Pauli matrices $\vec{\sigma}$. Then, after applying equation (3.2), this can be written as

$$\begin{aligned} I_3[\vec{A}] &= \hat{\mathcal{P}}\exp\left[\int_0^T d\tau\vec{A}(\tau)\frac{\partial}{\partial\vec{\rho}(\tau)}\right] \\ &\times\exp\left[-\frac{1}{2}\int_0^T d\tau_1 d\tau_2 G_F(\tau_1,\tau_2)\vec{\rho}(\tau_1)\vec{\rho}(\tau_2)\right]\exp\left(T\vec{\rho}_0\vec{\sigma}\right)\Bigg|_{\vec{\rho}=0}\,, \end{aligned} \tag{3.20}$$

where $T\rho_0^a = \int_0^T \rho^a(\tau)d\tau$. Again, since $\vec{\rho_0}$ is a Grassmann triplet, only terms up to order $(\rho_0)^3$ contribute in the expansion of the last exponential in (3.20), leaving us with[6]

$$
\begin{aligned}
I_3[\vec{A}] = \hat{\mathcal{P}} \exp\left[\int_0^T d\tau \vec{A}(\tau)\frac{\partial}{\partial\vec{\rho}(\tau)}\right] \exp\left[-\frac{1}{2}\int_0^T d\tau_1 d\tau_2\, G_F(\tau_1,\tau_2)\vec{\rho}(\tau_1)\vec{\rho}(\tau_2)\right] \times \\
\times \left[1 + T\vec{\rho_0}\vec{\sigma} - \frac{i\varepsilon^{abc}}{2!}T^2\rho_0^a\rho_0^b\sigma^c - \frac{i}{3!}\varepsilon_{abc}T^3\rho_0^a\rho_0^b\rho_0^c\right]\Bigg|_{\vec{\rho}=0}. \qquad (3.21)
\end{aligned}
$$

The expressions for I_3 and I_4 are in fact related. To see this, we note that the 't Hooft symbols $\eta^{\pm}_{a\mu\nu}$ [40] can be used to write

$$
\sigma_{\mu\nu} = i\begin{pmatrix} \eta^{+}_{a\mu\nu}\sigma_a & 0 \\ 0 & \eta^{-}_{a\mu\nu}\sigma_a \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (3.22)
$$

using an appropriate representation of the Euclidean Dirac matrices, and

$$
\begin{aligned}
\eta^{\pm}_{aij} &= \varepsilon_{aij}\,, \qquad & i,j &= 1,2,3\,, \\
\eta^{\pm}_{a4j} &= \pm\delta_{aj}\,, \qquad & j &= 1,2,3\,, \\
\eta^{\pm}_{a44} &= 0\,. & & \qquad (3.23)
\end{aligned}
$$

(Useful properties of the $\eta^{\pm}_{a\mu\nu}$ are given in [40].) As a consequence, when taking the trace over spinor indices of K in (1.4) we can write

$$
\begin{aligned}
\mathrm{tr}K(y,z;T) &= \sum_{\pm}\mathrm{tr}_\sigma \int_{x(0)=z}^{x(T)=y} \mathcal{D}x(\tau)\mathcal{P}\exp\int_0^T d\tau\left[-\mathcal{L}_{\pm}(x,\vec{\sigma})\right] \qquad (3.24) \\
&= 2\sum_{\pm}\int_{x(0)=z}^{x(T)=y} \mathcal{D}x(\tau)\hat{\mathcal{P}}\exp\int_0^T d\tau\left[-\mathcal{L}_{\pm}\left(x,\tfrac{\partial}{\partial\vec{\rho}}\right)d\tau\right] \times \qquad (3.25) \\
&\quad \times\exp\left[-\frac{1}{2}\int_0^T d\tau_1 d\tau_2\, G_F(\tau_1,\tau_2)\vec{\rho}(\tau_1)\vec{\rho}(\tau_2)\right]\left[1 - \frac{iT^3}{3!}\rho_0^a\rho_0^b\rho_0^c\varepsilon_{abc}\right]\Bigg|_{\rho=0},
\end{aligned}
$$

where

$$
\mathcal{L}_{\pm}(x,\vec{\sigma}) \equiv \frac{\dot{x}^2}{4} + i\dot{x}A + \frac{1}{2}\sigma_a\eta^{\pm}_{a\mu\nu}F_{\mu\nu} \pm \left(\sigma_a\eta^{\pm}_{a\mu\nu}\dot{x}^\mu A_5^\nu + i\partial A_5\right) + (D-2)A_5^2\,. \qquad (3.26)
$$

Equation (3.25) follows from (3.24) by direct application of (3.15) and (3.21). The functional trace of $K(T)$ can be found from (3.24) by setting $z = y$ and integrating over y, effectively imposing periodic boundary conditions on $x(\tau)$. Unlike in section 2, there is no Grassmann integral to be considered.

[6] Care must be taken here since the assumption that $\{\rho,\sigma\} = 0$ is inconsistent with the multiplication rule for the σ-matrices. Thus, it is important that the simplification of the SU(2) structure be carried out only after the functional derivatives with respect to ρ are allowed to act, or equivalently, after all σ matrices have been be ordered to the right of all ρ sources.

4. The VVA anomaly

Any new formalism for axial vector computations must, of course, be confronted with the axial anomaly. In [33] it was already shown that the present formalism correctly yields the anomalous divergence of the chiral current in the VVA function. Moreover, a complete cancellation between Feynman numerators and denominators was found in this calculation after a suitable integration by parts. We will now first extend that calculation to the off-shell case, and then rederive precisely the same integrand using the above $\mathrm{SU}(2)\times\mathrm{SU}(2)$ formulation.

4.1 The VVA anomaly from the master formula

As explained in [33] it is visible already at the path integral level that in the present formalism the anomalous divergence implied by the ABJ anomaly [46] is generally confined to the axial-vector currents. Calculation of the $VV\partial A$ amplitude either using the master formula eq. (2.21) or by a direct Wick-contraction of eq. (2.13) yields the following parameter integral,

$$
\begin{aligned}
k_3^\rho\langle A^\mu A^\nu A_5^\rho\rangle = 2\varepsilon^{\mu\nu\kappa\lambda}k_1^\kappa k_2^\lambda\int_0^\infty\frac{dT}{T}(4\pi T)^{-2}\prod_{i=1}^3\int_0^T d\tau_i\,\times\\
\times\exp\Big[\Big(G_{B12}-G_{B13}-G_{B23}\Big)k_1k_2-G_{B13}k_1^2-G_{B23}k_2^2\Big]\times\\
\times\Big\{(k_1+k_2)^2+(\dot G_{B12}+\dot G_{B23}+\dot G_{B31})(\dot G_{B13}-\dot G_{B23})k_1k_2-\\
-(\ddot G_{B13}+\ddot G_{B23})\Big\}
\end{aligned}
\tag{4.1}
$$

Here momentum conservation has been used to eliminate k_3. Removing the second derivatives $\ddot G_{B13}$ ($\ddot G_{B23}$) by a partial integration in τ_1 (τ_2), the expression in brackets turns into

$$
\begin{aligned}
k_1k_2\Big\{2-(\dot G_{B12}+\dot G_{B23}+\dot G_{B31})^2+\dot G_{B12}^2-\dot G_{B13}^2-\dot G_{B23}^2\Big\}+\\
+k_1^2(1-\dot G_{B13}^2)+k_2^2(1-\dot G_{B23}^2) =\\
=-\frac{4}{T}\Big[\Big(G_{B12}-G_{B13}-G_{B23}\Big)k_1k_2-G_{B13}k_1^2-G_{B23}k_2^2\Big].
\end{aligned}
\tag{4.2}
$$

In the last step we used the identities

$$
\begin{aligned}
\dot G_{Bij}+\dot G_{Bjk}+\dot G_{Bki} &= -\mathrm{sign}(\tau_i-\tau_j)\mathrm{sign}(\tau_j-\tau_k)\mathrm{sign}(\tau_k-\tau_i)\\
\dot G_{Bij}^2 &= 1-4\frac{G_{Bij}}{T}\,.
\end{aligned}
\tag{4.3}
$$

This is precisely the same expression which appears also in the exponential factor in (4.1). After performing the trivial T-integral we find therefore a complete cancellation between the Feynman numerator and denominator polynomials, and obtain without further integration the desired result,

$$
k_3^\rho\langle A^\mu A^\nu A_5^\rho\rangle = \frac{8}{(4\pi)^2}\varepsilon^{\mu\nu\kappa\lambda}k_1^\kappa k_2^\lambda\,. \tag{4.4}
$$

4.2 The VVA anomaly from $\mathrm{SU}(2) \times \mathrm{SU}(2)$

Alternatively we can compute this three-point function also using eq. (3.25). Evaluating the heat kernel in (3.25) with the background configuration

$$
\begin{aligned}
A^\mu(x) &= \varepsilon_1^\mu e^{ik_1 x} + \varepsilon_2^\mu e^{ik_2 x}\,, \\
A_5^\mu(x) &= \varepsilon_3^\mu e^{ik_3 x}\,,
\end{aligned}
\tag{4.5}
$$

and extracting the contribution which is linear in all three polarization vectors ε_j^μ we obtain

$$
\begin{aligned}
\operatorname{tr} K|_{VVA} = 2\sum_{\pm}\int_{x(0)=x(T)} & \mathcal{D}x(\tau) e^{-\int_0^T \frac{\dot{x}^2}{4}d\tau}\hat{\mathcal{P}}\times \\
&\times \int_0^T d\tau_1(-1)\left[i\dot{x}_{1\nu_1} + i\eta^{\pm}_{a_1\mu_1\nu_1}k_{1\mu_1}\frac{\partial}{\partial\rho_{a_1}(\tau_1)}\right]\varepsilon_1^{\nu_1}e^{ik_1x_1}\times \\
&\times \int_0^T d\tau_2(-1)\left[i\dot{x}_{2\nu_2} + i\eta^{\pm}_{a_2\mu_2\nu_2}k_{2\mu_2}\frac{\partial}{\partial\rho_{a_2}(\tau_2)}\right]\varepsilon_2^{\nu_2}e^{ik_2x_2}\times \\
&\times(\pm)\int_0^T d\tau_3(-1)\left[-k_{3\nu_2} + \eta^{\pm}_{a_3\mu_3\nu_3}\dot{x}_{3\mu_3}\frac{\partial}{\partial\rho_{a_3}(\tau_3)}\right]\varepsilon_3^{\nu_3}e^{ik_3x_3}\times \\
&\times \exp\left[-\frac{1}{2}\int_0^T d\tau_1 d\tau_2\, G_F(\tau_1,\tau_2)\vec{\rho}(\tau_1)\vec{\rho}(\tau_2)\right]\left[1-\frac{iT^3}{3!}\rho_0^a\rho_0^b\rho_0^c\varepsilon_{abc}\right]\Bigg|_{\rho=0} =
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
= -2\varepsilon_{1\nu_1}\varepsilon_{2\nu_2}\varepsilon_{3\nu_3}\sum_{\pm}(\pm)\int \mathcal{D}x\, e^{-\int_0^T \frac{\dot{x}^2}{4}d\tau}&\int_0^T d\tau_1 d\tau_2 d\tau_3 e^{i(k_1x_1+k_2x_2+k_3x_3)}\times \\
\times\Big[\dot{x}_{1\nu_1}\dot{x}_{2\nu_2}k_{3\nu_3} - \eta^{\pm}_{a\mu_1\nu_1}k_{1\mu_1}&\eta^{\pm}_{a\mu_2\nu_2}k_{2\mu_2}G_F(\tau_2,\tau_1)\mathrm{sign}(\tau_1-\tau_2)k_{3\nu_3} + \\
+\dot{x}_{1\nu_1}\eta^{\pm}_{a\mu_2\nu_2}k_{2\mu_2}&\eta^{\pm}_{a\mu_3\nu_3}\dot{x}_{3\mu_3}G_F(\tau_3,\tau_2)\mathrm{sign}(\tau_2-\tau_3) + (1\leftrightarrow 2) + \\
+\eta^{\pm}_{a\mu_1\nu_1}k_{1\mu_1}&\eta^{\pm}_{b\mu_2\nu_2}k_{2\mu_2}\eta^{\pm}_{c\mu_3\nu_3}\dot{x}_{3\mu_3}i\varepsilon_{cba}\mathrm{sign}(\tau_1,\tau_2,\tau_3)\Big]\,.
\end{aligned}
\tag{4.7}
$$

(Note the simplification $G_F(\tau,\tau')\mathrm{sign}(\tau'-\tau) = -1$.) Here the $\hat{\mathcal{P}}$ ordering of anticommuting variables has been resolved using the usual signum function as well as its following higher-point generalization

$$
\mathrm{sign}(\tau_1,\tau_2,\ldots,\tau_n) \equiv \prod_{1\le i<j\le n}\mathrm{sign}(\tau_i-\tau_j)\,. \tag{4.8}
$$

Further simplification is provided by the algebraic properties of the η symbols [40]. Of particular use are the identities

$$
\begin{aligned}
\eta^{\pm}_{a\mu_1\nu_1}\eta^{\pm}_{a\mu_2\nu_2} &= \delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2} - \delta_{\mu_1\nu_2}\delta_{\nu_1\mu_2} \mp \varepsilon_{\mu_1\nu_1\mu_2\nu_2}\,, \\
\varepsilon_{abc}\eta^{\pm}_{b\mu_2\nu_2}\eta^{\pm}_{c\mu_3\nu_3} &= \delta_{\mu_2\mu_3}\eta^{\pm}_{a\nu_2\nu_3} - \delta_{\mu_2\nu_3}\eta^{\pm}_{a\nu_2\mu_3} - \delta_{\nu_2\mu_3}\eta^{\pm}_{a\mu_2\nu_3} + \delta_{\nu_2\nu_3}\eta^{\pm}_{a\mu_2\mu_3}\,.
\end{aligned}
\tag{4.9}
$$

The application of these rules to an equation such as (4.7) can be greatly simplified if one recognizes that the rules for $\eta^+_{a\mu\nu}$ differ from those for $\eta^-_{a\mu\nu}$ only in the sign coefficient of $\varepsilon_{\mu\nu\rho\lambda}$. The effect of subtracting the $+$ and $-$ contributions is to double those terms which are proportional to the ε-tensor while canceling those which are not. Furthermore, for the ABJ anomaly, we are interested in the longitudinal axial polarization $\varepsilon_3 \propto k_3$ which together with conservation of momentum eliminates terms of the form $\varepsilon_{\mu_1\mu_2\nu_3\lambda}k_{1\mu_1}k_{2\mu_2}k_{3\nu_3}$.

We then integrate over the trajectories $x(\tau)$ using same contraction rules and path integral normalization as in section 2. This yields

$$\begin{aligned} k_{3\nu_3}\left\langle A_{\nu_1}(k_1)A_{\nu_2}(k_2)A_{5\nu_3}(k_3)\right\rangle = 2\varepsilon_{\nu_1\nu_2\mu_1\mu_2}k_1^{\mu_1}k_2^{\mu_2}\int_0^\infty \frac{dT}{T}(4\pi T)^{-2}\int_0^T d\tau_1 d\tau_2 d\tau_3 \times \\ \times \exp\left[k_1k_2G_{B12}+k_2k_3G_{B23}+k_1k_3G_{B13}\right]\times \\ \times\Big\{k_3^2+\left(\dot G_{B13}-\dot G_{B23}\right)\mathrm{sign}(\tau_1,\tau_2,\tau_3)k_1k_2- \\ -\left(\ddot G_{B13}+\ddot G_{B23}\right)\Big\} \end{aligned} \tag{4.10}$$

Using eq. (4.3) we see that this is precisely the same integrand as was obtained above in eq. (4.1).

5. inclusion of external fermions

The heat kernel evaluated by using the quantum mechanical path integral in conjunction with the FG approach to replacing Dirac matrices by a path integral over Grassmann variables can also be employed to compute Green's functions involving external fermions and quantized gauge fields. This we illustrate by considering the lagrangean

$$L=\overline{\psi}\left(\not p+\not A+\gamma_5\not A_5\right)\psi-\frac{1}{2}\left(\partial_\mu A_{5\nu}\right)\left(\partial^\mu A_5^\nu\right)-\frac{1}{2}\left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu\right) \tag{5.1}$$

in the Feynman gauge for an axial vector field $A_{5\mu}$, a vector field A_μ and a massless spinor field ψ. If we have an external component to the spinor field ψ, then the one-loop effective action involves the functional determinant.

$$\mathrm{sdet}^{-1/2}M=\mathrm{sdet}^{-1/2}\begin{pmatrix} 0 & \not p & \gamma_\nu & \gamma_5\gamma_\nu\psi \\ -\not p^T & 0 & -\left(\overline{\psi}\gamma_\nu\right)^T & -\left(\overline{\psi}\gamma_5\gamma_\nu\right)^T \\ \left(\gamma_\mu\psi\right)^T & \overline{\psi}\gamma_\mu & -p^2 & 0 \\ \left(\gamma_5\gamma_\mu\psi\right)^T & \overline{\psi}\gamma_5\gamma_\mu & 0 & -p^2 \end{pmatrix}. \tag{5.2}$$

Premultiplying (5.2) by

$$X=\begin{pmatrix} 0 & -\not p^T & 0 & 0 \\ \not p & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{5.3}$$

leads to

$$
\begin{aligned}
\mathrm{sdet}^{-1/2} XM = {} & \mathrm{sdet}^{-1/2}\begin{pmatrix} p^2 & 0 & \left(\overline{\psi}\gamma_\nu \not{p}\right)^T & \left(\overline{\psi}\gamma_5\gamma_\nu \not{p}\right)^T \\ 0 & p^2 & \not{p}\gamma_\nu\psi & \not{p}\gamma_5\gamma_\nu\psi \\ -\left(\gamma_\mu\psi\right)^T & -\overline{\psi}\gamma_\mu & p^2 & 0 \\ -\left(\gamma_5\gamma_\mu\psi\right)^T & -\overline{\psi}\gamma_5\gamma_\mu & 0 & p^2 \end{pmatrix} = \\
= {} & \mathrm{sdet}^{-1/2}\left\{\left[\begin{pmatrix} p_\rho & 0 & 0 & 0 \\ 0 & p_\rho & 0 & 0 \\ 0 & 0 & p_\rho & 0 \\ 0 & 0 & 0 & p_\rho \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & \left(\overline{\psi}\gamma_\nu\gamma_\rho\right)^T & \left(\overline{\psi}\gamma_5\gamma_\nu\gamma_\rho\right)^T \\ 0 & 0 & \gamma_\rho\gamma_\nu\psi & \gamma_\rho\gamma_5\gamma_\nu\psi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right]^2 + \right. \\
& +\frac{1}{2}\begin{pmatrix} 0 & 0 & i\left(\overline{\psi}\gamma_\nu\gamma_\rho\right)^T_{,\rho} & i\left(\overline{\psi}\gamma_5\gamma_\nu\gamma_\rho\right)^T_{,\rho} \\ 0 & 0 & -i\gamma_\rho\gamma_\nu\psi_{,\rho} & -i\gamma_\rho\gamma_5\gamma_\nu\psi_{,\rho} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \\
& \left. + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\left(\gamma_\mu\psi\right)^T & -\overline{\psi}\gamma_\mu & 0 & 0 \\ -\left(\gamma_5\gamma_\mu\psi\right)^T & -\overline{\psi}\gamma_5\gamma_\mu & 0 & 0 \end{pmatrix}\right\}.
\end{aligned}
\tag{5.4}
$$

This is of the form

$$\mathrm{sdet}^{-1/2}\left[(p-A)^2 + V\right] \tag{5.5}$$

provided

$$A_\rho = -\frac{1}{2}\begin{pmatrix} 0 & 0 & \left(\overline{\psi}\gamma_\nu\gamma_\rho\right)^T & \left(\overline{\psi}\gamma_5\gamma_\nu\gamma_\rho\right)^T \\ 0 & 0 & \gamma_\rho\gamma_\nu\psi & \gamma_\rho\gamma_5\gamma_\nu\psi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.6}$$

and

$$V = \begin{pmatrix} 0 & 0 & \frac{i}{2}\left(\overline{\psi}\gamma_\nu\gamma_\rho\right)^T_{,\rho} & \frac{i}{2}\left(\overline{\psi}\gamma_5\gamma_\nu\gamma_\rho\right)^T_{,\rho} \\ 0 & 0 & -\frac{i}{2}\gamma_\rho\gamma_\nu\psi_{,\rho} & -\frac{i}{2}\gamma_\rho\gamma_5\gamma_\nu\psi_{,\rho} \\ -\left(\gamma_\mu\psi\right)^T & -\overline{\psi}\gamma_\mu & 0 & 0 \\ -\left(\gamma_5\gamma_\mu\psi\right)^T & -\overline{\psi}\gamma_5\gamma_\mu & 0 & 0 \end{pmatrix}. \tag{5.7}$$

Since we can express

$$\ln \mathrm{sdet}^{-1/2}\left[(p-A)^2+V\right] = -\frac{1}{2}\mathrm{str}\ln\left[(p-A)^2+V\right] = \tag{5.8}$$
$$= \frac{1}{2\Gamma(s)}\int_0^\infty dT\, T^{s-1}\mathrm{str}\left\langle x\left|\exp -T\left[(p-A)^2+V\right]\right|y\right\rangle ,$$

where operator regularization [47] has been used,[7] we see that the one-loop effective potential can be worked out by considering the path integral for the matrix in (5.8)

$$M_{yz} = \mathcal{P}\int_{x(0)=z}^{x(T)=y}\mathcal{D}x(\tau)\exp\int_0^T d\tau\left[-\frac{\dot{x}^2}{4}+iA(x(\tau))\dot{x}(\tau)-V(x(\tau))\right] . \tag{5.9}$$

The FG procedure for replacing the γ matrices occurring in (5.6) and (5.7) when they are inserted into (5.9) is now straightforward; an example of how this is done in the context of the Yukawa model is given in [39].

6. Discussion

In this paper we have considered various aspects of handling γ-matrices that appear in the quantum mechanical path integral which arises when computing radiative effects in models in which a spinor is coupled to both a vector and axial vector field. These γ-matrices can either be replaced by an integral over a Grassmann parameter, or reduced to a "spin factor".

The Dirac matrix valued worldline lagrangean in (1.4) (and, indeed the squared operator which appears in equation (1.2)) commutes with γ_5 and leads to an apparent decoupling of the left and right-handed spin sectors, as we saw in equation (3.24). This property also led to simplifications in the calculation of the axial anomaly. In models with isotopic spin symmetry, SU(2) matrices τ_a enter into the path integral and can be treated in a manner identical to the way in which the Pauli spin matrices σ_a have been treated in (3.19)–(3.20). In fact, generators of the spinor representation of a group SO(N) can be treated in a manner similar to equations (1.4)–(1.5). This possibility was also considered in [48].

On the applied side, our main result is in the master formulas (2.18), (2.21). They constitute a generalization of the QED Bern-Kosower master formula for photon scattering. To discuss the potential usefulness of these formulas it is necessary to restate some known facts about the relation between the "string-inspired formalism" and standard field theory [10, 49, 13]. Generally, the parameter integrals obtained in string-inspired approaches correspond to a second order formulation of the field theory in question. For example, in scalar QED the parameter integral obtained for the N-photon amplitude from the Bern-Kosower master formula can be directly

[7] Of course one could as well use heat kernel regularisation.

identified with the corresponding Feynman parameter integral. This requires the rewriting of Feynman parameters as differences of proper-times, and the use of momentum conservation. The seagull vertex present in scalar QED then corresponds precisely to the δ-function appearing in the second derivative of the bosonic worldline Green's function.

The same identification works also for spinor QED, however not in its usual first-order formulation, but in a second-order formulation. The existence of such a reformulation of spinor QED has been known for a long time [50, 51]. However it has rarely been applied in practice, and was worked out in detail only recently in [52], where also the non-abelian generalization was investigated. Those second-order Feynman rules, as well as their generalization to the vector-axialvector case, can be directly read off eq. (1.2) and are given in appendix D. For the QED case they are an extension of the scalar QED rules, with a scalar propagator for the electron, and an additional vertex involving the $\sigma_{\mu\nu}$-matrix. (This is also the reason why in QED applications the "string-inspired" formalism generally allows one to obtain scalar QED results as a byproduct of spinor QED calculations [16, 22]).

Despite of this direct correspondence, the worldline representation of the parameter integrals has two advantages over the Feynman parameter representation. Firstly, by rewriting the integrand in terms of the "circular" functions G_B, G_F it is brought into a form where one can perform partial integrations in parameter space without generating boundary terms (in the abelian case considered here). In the QED case this can be used to remove all $\ddot{G}_B$'s appearing in the initial integrand, and to arrive at a Feynman numerator which is completely expressed in terms of the various $\dot{G}_{Bij}$, and homogeneous in the external momenta [10, 13]. This rearrangement leads to a canonical gauge invariant decomposition of the N-photon amplitude [53] and also to intriguing cancellations in the calculation of the QED 2-loop β-function [16]. Moreover it makes the worldline supersymmetry, implicit in the path integral, manifest at the parameter integral level [10, 13, 53] (for more on this point see appendix C). It will be interesting to work out the consequences of the partial integration procedure for the generalized master formulas. As we have already seen in section 4 it leads to a trivialization of the parameter integrals in the case of the ABJ anomaly.

A second advantage of the worldline representation is that the integrands are valid for an arbitrary ordering of the external legs, so that one can write down whole one-loop amplitudes "in one piece", with no necessity to sum over crossed diagrams. This property is less relevant at the one-loop level but leads to interesting consequences after multiloop generalization [15, 16].

Somewhat unusual is also the behaviour of the present formalism with respect to chiral symmetry in the dimensional continuation. It should be remembered that, in field theory, one has essentially a choice between two evils. If one preserves the anticommutation relation between γ_5 and the other Dirac matrices [54] then the chiral symmetry is preserved for parity-even fermion loops, but Dirac traces with an odd number of γ_5's are not unambiguously defined in general, requiring additional prescriptions. The main alternative is to use the 't Hooft–Veltman–Breitenlohner–Maison prescription [55, 56]. In this case there are no ambiguities, but the chiral symmetry is explicitly broken, so that in chiral gauge theories finite renormalizations generally become necessary to avoid violations of the gauge Ward identities [57, 58, 59].

Since our path integral representation was derived using an anticommuting γ_5, we have not broken the chiral symmetry. In particular, in the massless case the amplitude with an even number of axialvectors should coincide with the corresponding vector amplitude, and we have explicitly verified this fact for the two-point case. (Even though the structure of the resulting Feynman numerators is quite different from the equivalent ones derived from the ordinary Bern-Kosower master formula.) Nevertheless, we did not encounter any ambiguities even in the parity-odd case, not even in the anomaly calculation. As was explained already in part I, it is clear already from the structure of the vertex operators that only the axial vector currents can develop anomalous divergences. This property makes us optimistic about the potential usefulness of this formalism for calculations in chiral gauge theories.

Finally, let us mention some generalizations of the formalism developed here which would be entirely straightforward. First, the manipulations leading to eq. (1.6) could as well be carried out with an additional scalar background field (though in the massive case not with a pseudoscalar one). Then, also our restriction to the abelian case was largely a matter of expedience. Finally, the master formulas (2.18), (2.21) could be easily modified to take an additional external constant electromagnetic field into account along the lines of [22]. The special case of the 2-point vector–axial vector amplitude in a constant field was already considered in [26, 60]. In addition, one could also deal with the Lorentz violating theories considered in [61].

Acknowledgments

F.A.D. would like to thank Dmitri Gitman for pointing out reference [42]. C.S. thanks M. Reuter for various suggestions, and the Institute for Advanced Study, Princeton, for hospitality during the final stage of this work. We would like to thank the authors of [61], and [35] for having brought those references to our attention. NSERC of Canada provided financial support.

A. Coherent state representation of the effective action

In this appendix we use the coherent state formalism developed in [37] to transform the functional trace of the heat kernel $K(x, y; T)$ in eq. (1.4) into a quantum mechanical path integral. Our treatment closely parallels the one in [32] except that they work in six dimensions. Define matrices a_r^+ and a_r^-, $r = 1, 2$, by

$$a_1^\pm = \frac{1}{2}(\gamma_1 \pm i\gamma_3)\,, \qquad a_2^\pm = \frac{1}{2}(\gamma_2 \pm i\gamma_4)\,, \tag{A.1}$$

Those satisfy Fermi-Dirac anticommutation rules

$$\{a_r^+, a_s^-\} = \delta_{rs}\,, \qquad \{a_r^+, a_s^+\} = \{a_r^-, a_s^-\} = 0\,. \tag{A.2}$$

Thus we can use a_r^+ and a_r^- as creation and annihilation operators for a Hilbert space with a vacuum defined by

$$a_r^- |0\rangle = \langle 0 \mid a_r^+ = 0\,. \tag{A.3}$$

Next we introduce Grassmann variables η_r and $\bar\eta_r$, $r = 1, 2$, which anticommute with one another and with the operators $a_r^\pm$, and commute with the vacuum $\mid 0\rangle$. The coherent states are then defined as

$$\begin{aligned} \langle \eta \mid &\equiv i\langle 0 \mid (\eta_1 - a_1^-)(\eta_2 - a_2^-)\,, \qquad & \mid \eta\rangle &\equiv \exp(-\eta_1 a_1^+ - \eta_2 a_2^+) \mid 0\rangle\,, \\ \langle \bar\eta \mid &\equiv \langle 0 \mid \exp(-a_1^-\bar\eta_1 - a_2^-\bar\eta_2)\,, \qquad & \mid \bar\eta\rangle &\equiv i(\bar\eta_1 - a_1^+)(\bar\eta_2 - a_2^+) \mid 0\rangle\,. \end{aligned} \tag{A.4}$$

It is easily verified that those satisfy the defining equations for coherent states,

$$\begin{aligned} \langle \eta \mid a_r^- &= \langle \eta \mid \eta_r \qquad & a_r^- \mid \eta\rangle &= \eta_r \mid \eta\rangle\,, \qquad & \langle \eta \mid \bar\eta\rangle &= \exp(\eta_1\bar\eta_1 + \eta_2\bar\eta_2)\,, \\ \langle \bar\eta \mid a_r^+ &= \langle \bar\eta \mid \bar\eta_r \qquad & a_r^+ \mid \bar\eta\rangle &= \bar\eta_r \mid \bar\eta\rangle\,, \qquad & \langle \bar\eta \mid \eta\rangle &= \exp(\bar\eta_1\eta_1 + \bar\eta_2\eta_2)\,. \end{aligned} \tag{A.5}$$

Also one introduces the corresponding Grassmann integrals, defined by

$$\int \eta_i d\eta_i = \int \bar\eta_i d\bar\eta_i = i\,. \tag{A.6}$$

The $d\eta_r, d\bar\eta_r$ commute with one another and with the vacuum, and anticommute with all Grassmann variables and the $a_r^\pm$. This leads to the following completeness relation

$$\mathbb{1} = i\int \mid \eta\rangle\langle \eta \mid d^2\eta = -i\int d^2\bar\eta \mid \bar\eta\rangle\langle\bar\eta \mid\,, \tag{A.7}$$

($d^2\eta = d\eta_2 d\eta_1,\ d^2\bar\eta = d\bar\eta_1 d\bar\eta_2$), and to the following representation for a trace in the Fock space generated by the $a_r^\pm$,

$$\mathrm{Tr}(U) = i\int d^2\eta\,\langle -\eta \mid U \mid \eta\rangle\,. \tag{A.8}$$

We can now apply these fermionic coherent states together with the usual complete sets of coordinate states to rewrite the functional trace of (1.4) in the following way,

$$\begin{aligned} \mathrm{Tr} \mathrm{e}^{-T\Sigma} &= i\int d^4x \int d^2\eta\,\langle x, -\eta \mid \mathrm{e}^{-T\Sigma} \mid x, \eta\rangle \\ &= i^N \int \prod_{i=1}^N \Bigl(d^4x^i d^2\eta^i \langle x^i, \eta^i \mid \mathrm{e}^{-\frac{T}{N}\Sigma} \mid x^{i+1}, \eta^{i+1}\rangle \Bigr)\,, \end{aligned} \tag{A.9}$$

where $\Sigma = -(\partial_\mu + i\mathcal{A}_\mu)^2 + V$. The boundary conditions on the x and η integrations are $(x^{N+1}, \eta^{N+1}) = (x^1, -\eta^1)$. For the evaluation of this matrix element it will be useful to look first at the matrix elements of elementary products of Dirac matrices. For the product of two γ's one finds

$$\langle \eta^i \mid \gamma_\mu \gamma_\nu \mid \eta^{i+1} \rangle = -i \int d^2\bar{\eta}^{i,i+1} \langle \eta^i \mid \bar{\eta}^{i,i+1} \rangle \langle \bar{\eta}^{i,i+1} \mid \eta^{i+1} \rangle 2\, {}^i\psi_\mu \psi_\nu^{i+1} \,, \qquad \mu \neq \nu \,, \tag{A.10}$$

where

$$\begin{aligned} \psi_{1,2}^{i+1} &\equiv \frac{1}{\sqrt{2}}(\eta_{1,2}^{i+1} + \bar{\eta}_{1,2}^{i,i+1}) \,, \qquad & \psi_{3,4}^{i+1} &\equiv \frac{i}{\sqrt{2}}(\eta_{1,2}^{i+1} - \bar{\eta}_{1,2}^{i,i+1}) \,, \\ {}^i\psi_{1,2} &\equiv \frac{1}{\sqrt{2}}(\eta_{1,2}^{i} + \bar{\eta}_{1,2}^{i,i+1}) \,, \qquad & {}^i\psi_{3,4} &\equiv \frac{i}{\sqrt{2}}(\eta_{1,2}^{i} - \bar{\eta}_{1,2}^{i,i+1}) \,. \end{aligned} \tag{A.11}$$

To verify this equation one rewrites the Dirac matrices in terms of the $a_r^{\pm}$ and then inserts a complete set of coherent states $\mid \bar{\eta}^{i,i+1} \rangle$ in between them.

To take also the γ_5 - matrix into account it is crucial to observe that, expressed in terms of the $a_r^{\pm}$, it is identical to the fermion number counter or "G-parity operator" $(-1)^F$ [4],

$$\gamma_5 = (-1)^F = (1 - 2F_1)(1 - 2F_2) \,, \tag{A.12}$$

where

$$F \equiv F_1 + F_2, \qquad \text{with} \qquad F_i = a_i^+ a_i^- \,. \tag{A.13}$$

From the identity

$$\langle -\eta \mid (-)^F = i\langle 0 \mid \prod_{r=1}^{2} (-\eta_r - a_r^-)(1 - 2a_r^+ a_r^-) = i\langle 0 \mid \prod_{r=1}^{2} (-\eta_r + a_r^-) = \langle \eta \mid \tag{A.14}$$

it is clear that the presence of $(-1)^F$ can be taken into account by switching the boundary conditions on the Grassmann path integral from antiperiodic to periodic. With this information it is now easy to compute that

$$\begin{aligned} &\langle x^i, \eta^i \mid \mathrm{e}^{-T\Sigma[p,A,A_5,\gamma_\mu\gamma_\nu,\gamma_5]} \mid x^{i+1}, \eta^{i+1} \rangle = \\ &= -\frac{i}{(2\pi)^4} \int d^4 p^{i,i+1} d^2\bar{\eta}^{i,i+1} \mathrm{e}^{i(x^i - x^{i+1})p^{i+1} + (\eta^i - \eta^{i+1})_r \bar{\eta}_r^{i,i+1}} \times \\ &\times \left[1 - \frac{T}{N} \Sigma \left[p^{i,i+1}, A^{i,i+1}, A_5^{i,i+1}, 2\, {}^i\psi_\mu \psi_\nu^{i+1}, (-1)^F \right] + O\left(\frac{T^2}{N^2} \right) \right] . \end{aligned} \tag{A.15}$$

Here the superscript $\phi^{i,i+1}$ on a field denotes the average of the corresponding fields with superscripts i and $i+1$. Inserting this result back into eq. (A.9) one obtains, after symmetrizing the positions of the Grassmann variables in the exponentials

$$\begin{aligned} \mathrm{Tr}\mathrm{e}^{-T\Sigma} = &\frac{1}{(2\pi)^{4N}} \int \prod_{i=1}^{N} d^4x^i d^4p^{i,i+1} d^2\eta^i d^2\bar{\eta}^{i,i+1} \left(1 - \frac{T}{N}\Sigma_i + \mathrm{O}\left(\frac{T^2}{N^2}\right) \right) \times \\ &\times \exp\left\{ \sum_{i=1}^{N} \left[i(x^i - x^{i+1})p^{i,i+1} + \frac{1}{2}(\eta_r^i - \eta_r^{i+1})\bar{\eta}_r^{i,i+1} - \frac{1}{2}\eta_r^i(\bar{\eta}_r^{i-1,i} - \bar{\eta}_r^{i,i+1}) \right] \right\} . \end{aligned} \tag{A.16}$$

Introducing an interpolating proper-time τ such that $\tau_1 = T$, $\tau^{N+1} = 0$, and $\tau^i - \tau^{i+1} = \frac{T}{N}$, and taking the limit $N \to \infty$ in the usual naive way, we finally obtain the following path integral representation,

$$
\begin{aligned}
\mathrm{Tr} e^{-T\Sigma} &= \int \mathcal{D}p \int \mathcal{D}x \int_P \mathcal{D}\eta \mathcal{D}\bar{\eta} \times \\
&\times \exp\biggl\{ \int_0^T d\tau \Bigl[i\dot{x}p + \frac{1}{2}\dot{\eta}_r\bar{\eta}_r - \frac{1}{2}\eta_r\dot{\bar{\eta}}_r - \Sigma[p, A, A_5, 2\psi_\mu\psi_\nu, \hat{\gamma}_5] \Bigr] \biggr\}
\end{aligned}
\tag{A.17}
$$

The "P" denotes the antiperiodic boundary conditions which we have for $\eta, \bar{\eta}$ originally. The operator $(-1)^F$ has turned into an operator $\hat{\gamma}_5$ whose only raison d' être is to determine the boundary conditions of the Grassmann path integral; after expansion of the interaction exponential a given term will have to be evaluated using antiperiodic (periodic) boundary conditions on $\mathcal{D}\eta$, $\mathcal{D}\bar{\eta}$, if it contains $\hat{\gamma}_5$ at an even (odd) power. After the boundary conditions are determined $\hat{\gamma}_5$ can be replaced by unity. The continuum limit of eqs. (A.11) is

$$
\psi_{1,2}(\tau) = \frac{1}{\sqrt{2}}(\eta_{1,2}(\tau) + \bar{\eta}_{1,2}(\tau)), \qquad \psi_{3,4}(\tau) = \frac{i}{\sqrt{2}}(\eta_{1,2}(\tau) - \bar{\eta}_{1,2}(\tau)). \tag{A.18}
$$

This suggests a change of variables from $\eta, \bar{\eta}$ to ψ, which we complete by rewriting the fermionic kinetic term,

$$
\frac{1}{2}\dot{\eta}_r\bar{\eta}_r - \frac{1}{2}\eta_r\dot{\bar{\eta}}_r = -\frac{1}{2}\psi_\mu\dot{\psi}_\mu . \tag{A.19}
$$

The boundary conditions are now $(x(T), \psi(T)) = (x(0), -\psi(0))$. Finally, we note that the momentum path integral is Gaussian, and perform it by a naive completion of the square (for a less unscrupulous treatment of this point see again [32], as well as for the various normalization factors involved). This brings us to our following final result[8]

$$
\begin{aligned}
\Gamma_{\mathrm{spin}}[A, A_5] &= -2 \int_0^\infty \frac{dT}{T} \mathrm{e}^{-m^2 T} \int \mathcal{D}x \int \mathcal{D}\psi \, \mathrm{e}^{-\int_0^T d\tau \, L(\tau)} \\
L(\tau) &= \frac{1}{4}\dot{x}^2 + \frac{1}{2}\psi_\mu\dot{\psi}^\mu + i\dot{x}^\mu A_\mu - i\psi^\mu F_{\mu\nu}\psi^\nu - \\
&\quad -2i\hat{\gamma}_5\dot{x}^\mu\psi_\mu\psi_\nu A_5^\nu + i\hat{\gamma}_5\partial_\mu A_5^\mu + (D-2)A_5^2 .
\end{aligned}
\tag{A.20}
$$

B. Wick's theorem and the spin path integral

In this appendix we discuss the connection between Wick's theorem (3.2) as it is used in (3.3) and the Feynman rules of the spin path integral (1.6) as described in section 2, where one is generally interested in the trace of the heat kernel

$$
\int d^D y \, \mathrm{tr} \, K(y, y; T) .
$$

[8] Our definition of the Euclidean effective action differs by a sign from the one used in [32].

Expanding (3.3) in powers of its axialvector (and, in the more general case, pseudoscalar) couplings (i.e in powers of γ_5), we notice that there are only two possible nonzero traces which can arise.

For *even* powers in the γ_5 expansion we have

$$\mathrm{tr}\exp\left(T\rho_0\gamma\right) = 4\,. \tag{B.1}$$

In this case, the partial derivatives $\frac{\partial}{\partial\rho}$ which occur in (3.3) are forced to act on the propagator exponential and lead to Feynman contractions which resemble (2.5)

$$\langle\psi^\mu(\tau_1)\psi^\nu(\tau_2)\rangle = \frac{1}{2}\mathrm{sign}(\tau_1-\tau_2)g^{\mu\nu}\,,$$

where we make the correspondence $\frac{\partial}{\partial\rho^\mu(\tau)} \to \sqrt{2}\psi^\mu(\tau)$.

For *odd* powers we have

$$\mathrm{tr}\,\gamma_5\exp\left(T\rho_0\gamma\right) = \frac{4T^4}{4!}\varepsilon^{\mu\nu\lambda\rho}\rho_0^\mu\rho_0^\nu\rho_0^\lambda\rho_0^\rho\,. \tag{B.2}$$

For such processes, the zero-mode contributions to the propagator exponent are prohibited from contributing, due to the Grassmann character of ρ_0. As a result, we may safely remove the zero modes from the exponential, replacing it with

$$\exp\left[-\frac{1}{2}\int_0^T d\tau_1 d\tau_2\left(\mathrm{sign}(\tau_1-\tau_2)-2\frac{\tau_1-\tau_2}{T}\right)\rho(\tau_1)\rho(\tau_2)\right]\,. \tag{B.3}$$

So that the partial derivatives $\frac{\partial}{\partial\rho}$ may act either on the zero mode terms or the propagator exponential (B.3). Making the correspondence $\frac{\partial}{\partial\rho^\mu(\tau)} \to \sqrt{2}\left[\psi_0^\mu+\xi^\mu(\tau)\right]$ we arrive at a Feynman rule similar to the zero-mode rule in (2.4)

$$\int d^4\psi_0\,\psi_0^\mu\psi_0^\nu\psi_0^\kappa\psi_0^\lambda = \varepsilon^{\mu\nu\kappa\lambda}$$

and one similar to the non-zero mode Feynman rule in (2.5)

$$\langle\xi^\mu(\tau_1)\xi^\nu(\tau_2)\rangle = \frac{1}{2}\left(\mathrm{sign}(\tau_1-\tau_2)-2\frac{\tau_1-\tau_2}{T}\right)g^{\mu\nu}\,.$$

C. A comment on worldline supersymmetry

It is of interest to consider the supersymmetry present in the coherent state formulation of the quantum mechanical path integral in the case where there is only an external vector field (and no background axial or spinor fields). These considerations will demonstrate the importance of having anti-periodic boundary conditions for the Grassmann variables being integrated.

It is known that the supersymmetry properties of the worldsheet string action put strong restrictions on the cycle structure of the worldline approach to amplitude calculations. In fact, the kinematic factors must vanish when the Bosonic and Fermionic Green's functions both satisfy periodic boundary conditions [62, 10].

Here, we show how a simplified worldline supersymmetry can be used to arrive at the same conclusion without resorting to any analysis based on the cyclic structure of the Green's functions.

We begin with the following path integral expression

$$K_{\pm} = \int \mathcal{D}x(\tau) \int \mathcal{D}\psi(\tau) e^{-\int_0^T L(\tau)d\tau}\,, \tag{C.1}$$

where L is given in equation (1.7) (with $A_5 = 0$), $x(\tau)$ is a closed Bosonic path, and $\psi(\tau)$ is a Fermionic trajectory with the boundary condition $\psi(T) = \pm\psi(0)$.

The one-loop effective action for the vector field A is well-known to be related to K_-, but if, instead we consider the path integral K_+ with *periodic* boundary conditions on the $\psi(\tau)$ variable, then we notice that it possesses the following supersymmetry

$$\begin{aligned} Qx^\mu &= -2\psi^\mu\,, \\ Q\psi^\mu &= \dot{x}^\mu\,. \end{aligned} \tag{C.2}$$

It is trivial to show that the supersymmetry generator has the property

$$Q^2 = -2\frac{d}{d\tau}\,. \tag{C.3}$$

We now observe that the lagrangean L is in the image of the supersymmetry generator, specifically,

$$L = Q\left[\tfrac{1}{4}\dot{x}\psi + i\psi A\right]\,, \tag{C.4}$$

which, by (C.3) is sufficient to show that $\int_0^T d\tau L$ is supersymmetric as long as $x(\tau)$ and $\psi(\tau)$ are both periodic. We define $\Omega = \int_0^T d\tau\left(\tfrac{1}{4}\dot{x}\psi + i\psi A\right)$.

We will now demonstrate, using the techniques of [63], that a supersymmetric Ward identity guarantees that K_+ is independent of the vector field A.

The functional derivative of K_+ with respect to $A^\mu(x)$ is

$$\frac{\delta}{\delta A^\mu(x)}K_+ = -\int \mathcal{D}x(\tau)\int D\psi(\tau)e^{-Q\Omega}Q\Omega_\mu(x)\,, \tag{C.5}$$

where $\Omega_\mu(x) = \frac{\delta\Omega}{\delta A^\mu(x)}$.

Using the standard path-integral derivation of Ward identities, we now consider the path integral

$$\int \mathcal{D}x(\tau)\int \mathcal{D}\psi(\tau)e^{-Q\Omega}\Omega_\mu(x)\,, \tag{C.6}$$

which should be invariant under the following change of variables generated by (C.2)

$$
\begin{aligned}
x'^{\mu} &= x^{\mu} + \epsilon Q x^{\mu}\,, \\
\psi'^{\mu} &= \psi^{\mu} + \epsilon Q \psi^{\mu}\,, \\
\Omega' &= \Omega + \epsilon Q \Omega\,,
\end{aligned}
\tag{C.7}
$$

where ϵ is some Grassmann number. From (C.2) it is straightforward to see that this transformation preserves the periodic boundary conditions of $x(\tau)$ and $\psi(\tau)$ and that its Jacobian superdeterminant is unity. The action $Q\Omega$ is invariant under (C.7). Hence, we discover that (C.6) is equal to

$$
\int \mathcal{D}x(\tau) \int \mathcal{D}\psi(\tau) e^{-Q\Omega} \left[\Omega_{\mu}(x) + \epsilon Q \Omega_{\mu}(x)\right] . \tag{C.8}
$$

The equality of (C.6) and (C.8) immediately implies that (C.5) vanishes and that K_+ is perturbatively independent of $A^{\mu}(x)$.

Considering the simplicity of this proof, the topological nature of these amplitudes is surprisingly nontrivial to see at the parameter integral level. Things work out neatly for the case where only a constant background field is present. Treating this case in Fock-Schwinger gauge along the lines of [64] one finds that the path integral determinant factors into two Euler-Heisenberg type determinants from $\int \mathcal{D}y$ and $\int \mathcal{D}\xi$, which cancel by supersymmetry, and a Grassmann zero mode contribution. This zero mode contribution is given, for $D = 4n$, by the n-th Pontrijagin density, thus leading to a topological number indeed (which vanishes for the abelian case). However things are less transparent for the full N-point amplitude. Here the main term, not involving the Grassmann zero mode, can be seen to vanish after applying the partial integration procedure and the appropriate Bern-Kosower substitution rule [53] (which originally was derived in the string-context from worldsheet supersymmetry [10]). However there exist many more contributions involving the zero mode, and the mechanism of their vanishing seems not obvious.

D. Second-order Feynman rules

Here we give the second-order (Euclidean) Feynman rules for the lagrangean (5.1), as following from eq. (1.2). We have now restituted the coupling constants, $A \to eA, A_5 \to e_5 A_5$. The photon propagator is the usual one.

A factor $-\frac{1}{2}$ is to be supplied for a closed fermion loop, as well as a spinor trace (even for diagrams not involving Dirac matrices, which then carry the unit matrix). For more information on the second order formalism see [52]. Note that those rules remain valid in dimensional regularization with no further changes; there is no analogue of the D-dependent coefficient which we had in the worldline lagrangean.

The calculation of the one-loop amplitude for M vectors and N axialvectors with these rules yields, using Feynman parameters, precisely the same integrands as we

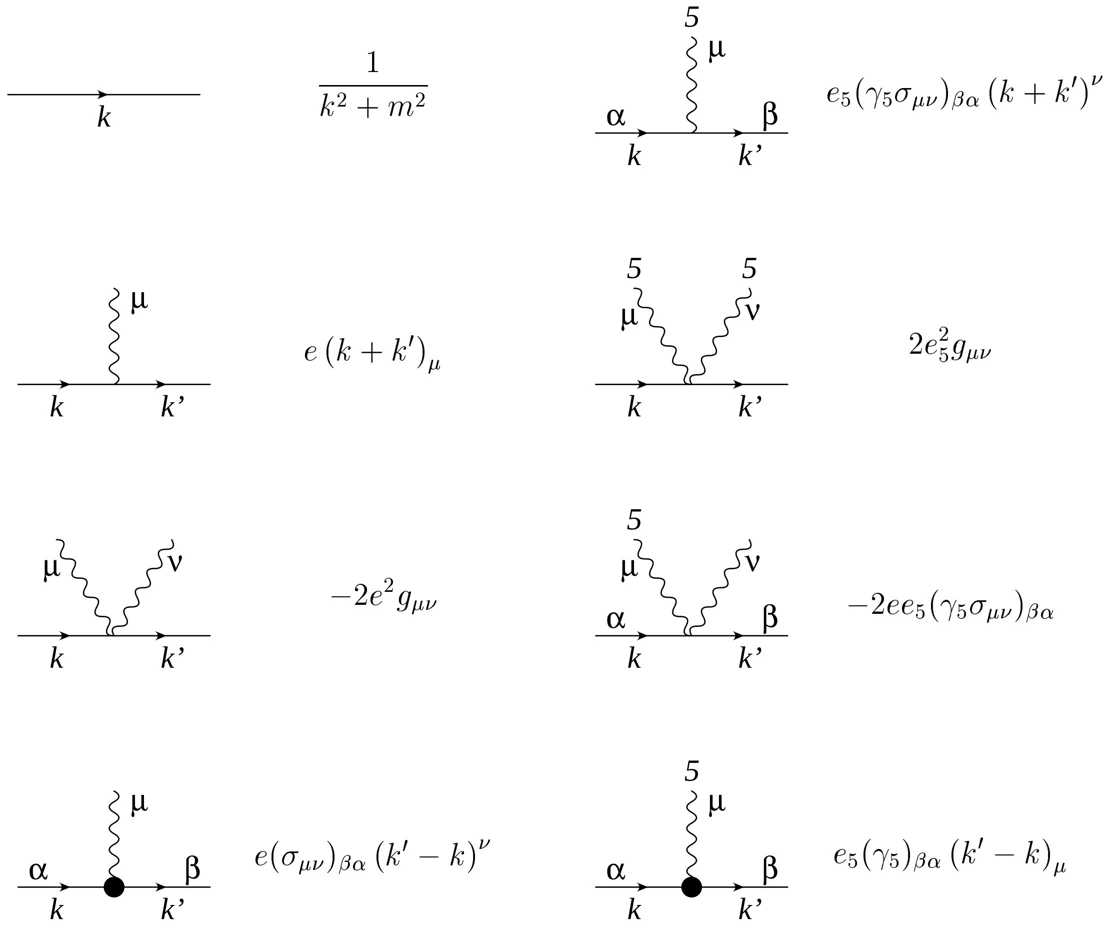

Figure 1: Second order Feynman rules

have in our master formulas eqs. (2.18), (2.21). However the master formulas already give the complete amplitude, with all crossed diagrams included. An individual diagram is represented by an ordered sector $\tau_{i_1} \geq \tau_{i_2} \geq s$, the Feynman parameters being the differences of adjacent τ_{i_j}-parameters (with our conventions the charge flow is in the direction of *ascending* τ_{i_j}).

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